Multiscale analysis of information correlations in an infinite-range, ferromagnetic Ising system

S. Gheorghiu-Svirschevski and Y. Bar-Yam
New England Complex Systems Institute, Cambridge, Massachusetts 02138, USA

(Received 26 July 2003; revised manuscript received 12 July 2004; published 7 December 2004)

The scale-specific information content of the infinite-range, ferromagnetic Ising model is examined by means of information-theoretic measures of high-order correlations in finite-sized systems. The order-disorder transition region can be identified through the appearance of collective order in the ferromagnetic phase. In addition, it is found that near the transition, the ferromagnetic phase is marked by characteristic information oscillations at scales comparable to the system size. The amplitude of these oscillations increases with the total number of spins, so that large-scale information measures of correlations are nonanalytic in the thermodynamic limit. In contrast, correlations at scales small relative to the system size have a monotonic behavior both above and below the transition point, and a well-defined thermodynamic limit.

DOI: 10.1103/PhysRevE.70.066115 PACS number(s): 89.75.Fb, 89.70.+c

I. INTRODUCTION

The emergence of order and structure in physical systems is conventionally quantified by means of various statistical mechanical measures, such as specific heat, susceptibilities, correlation functions, correlation lengths, and structure factors [1]. Information theory [2] may provide alternative tools that can extend and strengthen the statistical measures, especially for application to the study of complex systems or related models, such as neural networks [3], cellular automata [4], or chaotic dynamical systems [5]. In turn, the suitability of these new tools is best tested against the canonical models of statistical physics for structure-generating processes, particularly of critical phenomena.

Information theory has most often been associated with a variety of entropy/information-related measures [6] of the entire phase space. However, information theory as a characterization of the relationship between components, generalizing the study of correlations, has been less well studied. With this perspective in mind, we present in this paper a study of k-fold information correlations in the well-known ferromagnetic, infinite-range Ising model. Just as the pair correlations of a system are related to its dynamic response, higher-order correlations may also be relevant to the response of a complex system under external coherent interactions [8]. Here our general intention is to examine subsystem degrees of freedom of such complex systems in relation to the emergence of an order parameter in the thermodynamic limit. To this end, we apply a measure of the phase-space volume of k-fold correlations defined [9] as the amount of information (complexity) $C(k)$ that can be obtained through observations of k or more spins. k measures scale as the number of correlated spins rather than distance. $C(k)$ is related to the amount of information $D(k)$ obtained from sets of exactly k spins as $D(k)=C(k)-C(k+1)$. For a set of N Ising spins with a statistics described by a probability distribution $P(\{s_i\})$, $\{s_i\}\equiv\{s_1,s_2,\ldots,s_N\}$, the complexity $C_N(k)$ at the (integer) *scale* k is given by

$$C_N(k)=\sum_{j=0}^{k-1}(-1)^{k-1-j}\binom{N-j-1}{k-j-1}Q(N-j), \tag{1}$$

where $Q(N-j)$ represents the information content (Shannon entropy) of all possible subsets of $(N-j)$ spins,

$$Q(N-j)=-\sum_{\{i_1,i_2,\ldots,i_j\}}\sum_{\{s_i\}}P(\{s_i\})\log_2\sum_{s_{i_1},s_{i_2},\ldots,s_{i_j}}P(\{s_i\}). \tag{2}$$

We note that this measure was developed also in Ref. [10], in close analogy with Green's expansion for the entropy in the statistical mechanics of fluids [11], although the above explicit expressions are not given there. This formalism is quite unlike the conventional information measures on character strings [2], which have motivated information theory applications in physical systems confined to one dimension where thermodynamic phase transitions do not occur [7].

We examined the complexity (1) both numerically and analytically for an infinite-range Ising ferromagnet in thermal equilibrium, described, as usual, by a canonical distribution

$$P(\{s_i\})=\frac{1}{Z}\exp\left[\beta\frac{J}{N}\sum_{i>j=1}^{N}s_is_j\right]. \tag{3}$$

Here the energy functional in the exponential is scaled by the size N of the system to ensure a well-behaved thermodynamic limit $N\to\infty$, and the coupling constant is positive, $J\geqslant 0$. Although the permutational symmetry of the model reduces distribution (3) to a simple functional of the total magnetization $M=\sum_{i=1}^{N}s_i$ (as $P(M)\sim\exp[(\beta J/2N)M^2]$), neither the information of subsystems $Q(N-j)$ [Eq. (2)] nor the complexities $C_N(k)$ [Eq. (1)] can be given simple closed-form expressions. This is in contrast to common thermodynamic quantities, which have well-known expressions.

Since the mean-field approximation is known to be the exact solution for this system in the thermodynamic limit, we will compare results for the exact model to corresponding mean-field values. The standard mean-field approximation is, however, a *decoupling* approximation, which provides a simple working framework precisely by neglecting explicit correlations between spins. Thus, in this case, the correlation measure $C_N(k)$ vanishes identically on all scales, except on the finest one, $k=1$, for which it equals the total entropy. To avoid this oversimplification, we amend the usual mean-field approximation by accounting explicitly for the two equivalent, but macroscopically distinct, orientations of the average magnetization. The corresponding probability distribution is written

$$P^{(mf)}(\{s_i\}) = \frac{1}{2}\left[\prod_{i=1}^{N}\frac{1+ms_i}{2} + \prod_{i=1}^{N}\frac{1-ms_i}{2}\right], \qquad (4)$$

where the average magnetization per spin m is given by either solution of the mean-field self-consistency equation

$$m = \tanh(\beta J m). \qquad (5)$$

The probability distribution given in Eq. (4) generates essentially the same thermodynamics as the usual mean-field ansatz, but avoids simple factorization, and the implicit loss of larger-scale complexity. Thus it provides a nontrivial, albeit not necessarily accurate, description of spin correlations in the ferromagnetic phase transition.

Our numerical studies of information correlations in the ordered and disordered phases are presented and discussed in Sec. II. In Sec. III, we detail the analytical apparatus behind our modified mean-field approach [Eq. (4)], leading to a closed-form approximation for fine-scale, mean-field complexities. In Sec. IV, we present a refinement of the mean-field model which brings the corresponding results significantly closer to the exact model. Concluding remarks are provided in Sec. V.

II. NUMERICAL RESULTS

The calculations were performed on systems with up to $N=20$ spins. Higher values of N are possible but do not provide additional insights for the analysis described here. For each system size, the complexities $C_N(k)$ were calculated throughout the entire scale range $1 \leqslant k \leqslant N$, for coupling strengths in the range $0 \leqslant \beta J \leqslant 4$. The upper limit of the interaction range is large enough to describe the asymptotic strong-coupling regime.

The finite size of the systems studied results in an observable difference in the physics of the exact model as compared to the mean-field counterpart. While the mean-field model is *defined* by means of an order parameter that exhibits a well-defined phase transition regardless of the system size, the exact model displays a true phase transition only in the thermodynamic limit. Nevertheless, a reasonably well-defined change of regime can be seen also for the finite-size models, through the behavior of k-fold correlations. This behavior reflects the emergence of the order parameter in the thermodynamic limit of the model.

We will present results for $C(k>1)$. $C(1)$ is the system entropy and has a well-characterized behavior in both the exact and the mean-field model. It can be obtained as [9]

$$C(1) = N - \sum_{k=2}^{N} C_N(k).$$

We also show in the Appendix that for the modified mean-field model the entropy differs from the usual mean-field entropy by a small quantity (less than one bit), which vanishes in the thermodynamic limit.

We begin by examining the coupling strength dependence of complexities at all scales for systems of various sizes (Fig. 1). The disordered phase of the mean-field model, below the transition point at $(\beta J)_{\rm cr}=1$, samples a uniform distribution of configurations, and therefore shows no correlations at any scale $k>1[C_N(k\geqslant 2)=0]$. In the exact model, this interaction range corresponds to a weak-coupling regime, and a largely uncorrelated disordered phase, with low complexities for $k>1$.We observe that significant correlations are restricted to fine scales ($k \ll N$), and vanish as the coupling decreases. Large-scale correlations are virtually absent over the entire range $\beta J \leqslant 1$. In both the exact and the mean-field model, large-scale correlations develop with significant amplitude for coupligs $\beta J \geqslant 1$. For the mean-field model, this marks a sharp ferromagnetic phase transition. In the strong-coupling limit, in the ferromagnetic phase, both the exact and the mean-field complexities approach 1 asymptotically at every scale. This unit complexity is the single bit of information necessary to describe the two possible collective states, and confirms the alignment of all spins along one of two macroscopic directions.

Remarkably, for $\beta J>1$ we note a striking difference in the behavior of fine-scale and large-scale complexities with increasing interaction strength. Fine-scale complexities, for $k \ll N$, increase gradually and mostly monotonically with the strength of the interaction, until they saturate under sufficiently strong coupling. In contrast, large-scale correlations, as measured by complexities at scales k comparable to the system size $[1-(k/N) \ll 1]$, show a strongly nonmonotonic variation. In a somewhat counterintuitive fashion, some complexities are seen to become (strongly) negative. The phenomenon was found in Ref. [9] to be a signature of mutual dependencies between multiple (spin) variables that are not described by the dependencies between smaller subgroups of spins. It is, for example, found in the case of a frustrated antiferromagnet where the frustration occurs for groups of three spins but not for two. Here the collective "constraint" seems to have its origin in the gradual "localization" of the statistically populated spin-configurations around the fully aligned states, and the concomitant "exclusion" of configurations with low total magnetization. The behavior is much more dramatic, with significantly larger positive and negative excursions, for the mean-field model because it imposes the constraints on collective magnetization more directly. The behavior disappears in the strong-coupling limit because pairwise interactions are strong enough to force alignment, not just multiple interactions. (Intuitively, this corresponds to the behavior of finite correlation lengths in a spatial Ising

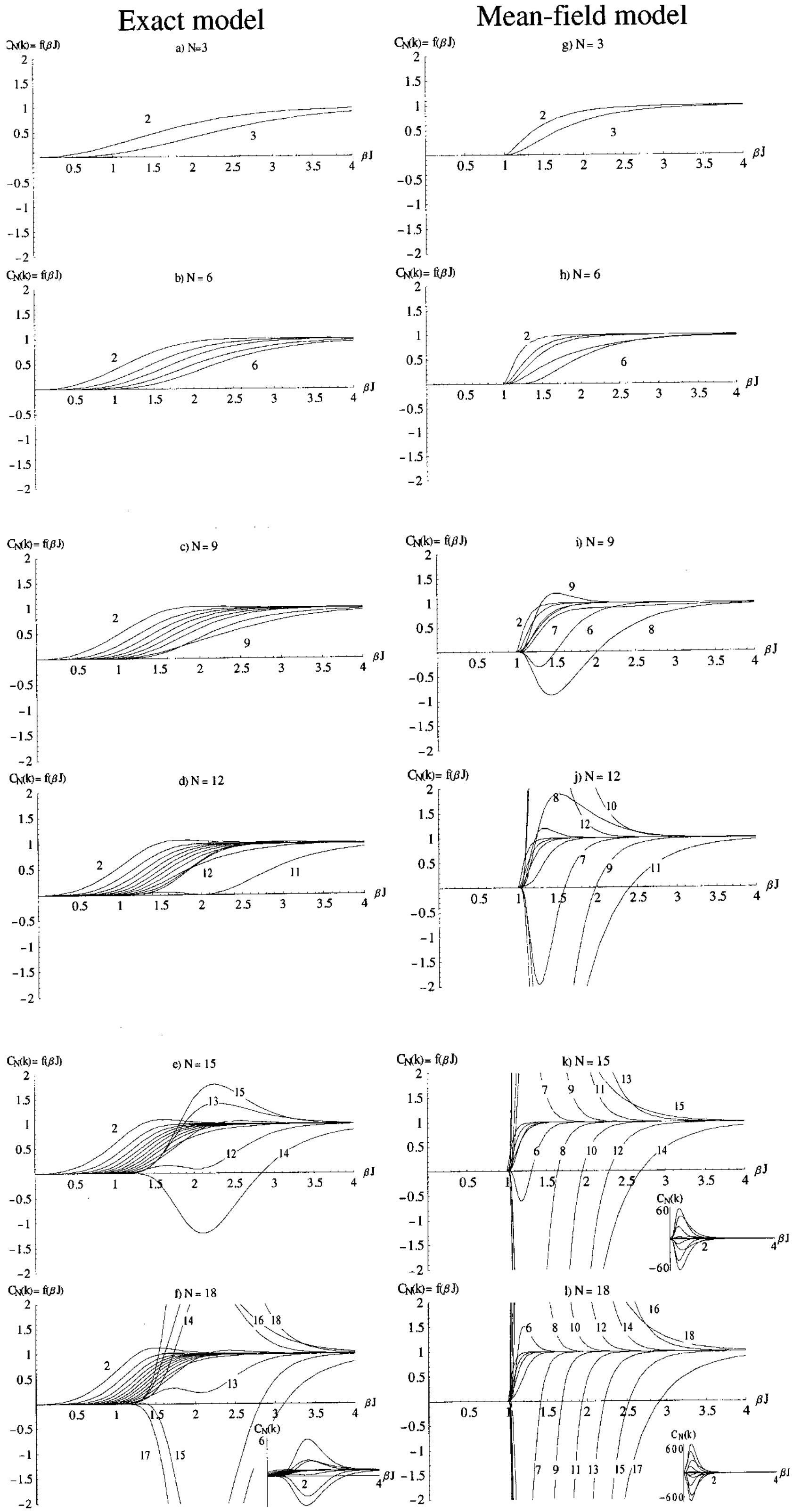

FIG. 1. The complexities $C_N(k)$ of a system of $N=3,6,9,12,15$, or 18 spins for increasing scales $k=2,3,\ldots,N$, as a function of the coupling strength βJ. Exact infinite-range Ising model results are shown in (a)–(f) (left) panels, modified mean-field model results in (g)–(l) (right) panels. Left and right panels compare exact and mean-field results in this and Figs. 2–4.

model, where in the ordered phase, disorder persists on scales shorter than the correlation length.)

The different behavior of fine- and large-scale correlations becomes more clearly visible when the complexities for systems of a given size are plotted against the scale k at various values of the coupling strength [Figs. 2(a)–2(d)]. In the ferromagnetic phase of the mean-field model [couplings $\beta J \geqslant (\beta J)_{\mathrm{cr}}=1.0$], the large-scale complexities display a characteristic oscillatory behavior, whose amplitude increases rapidly with the system size. A similar oscillatory pattern is seen in the exact model with more than $\sim$15 spins, under couplings $\beta J \geqslant 1.5$. The amplitude of the oscillations also in-

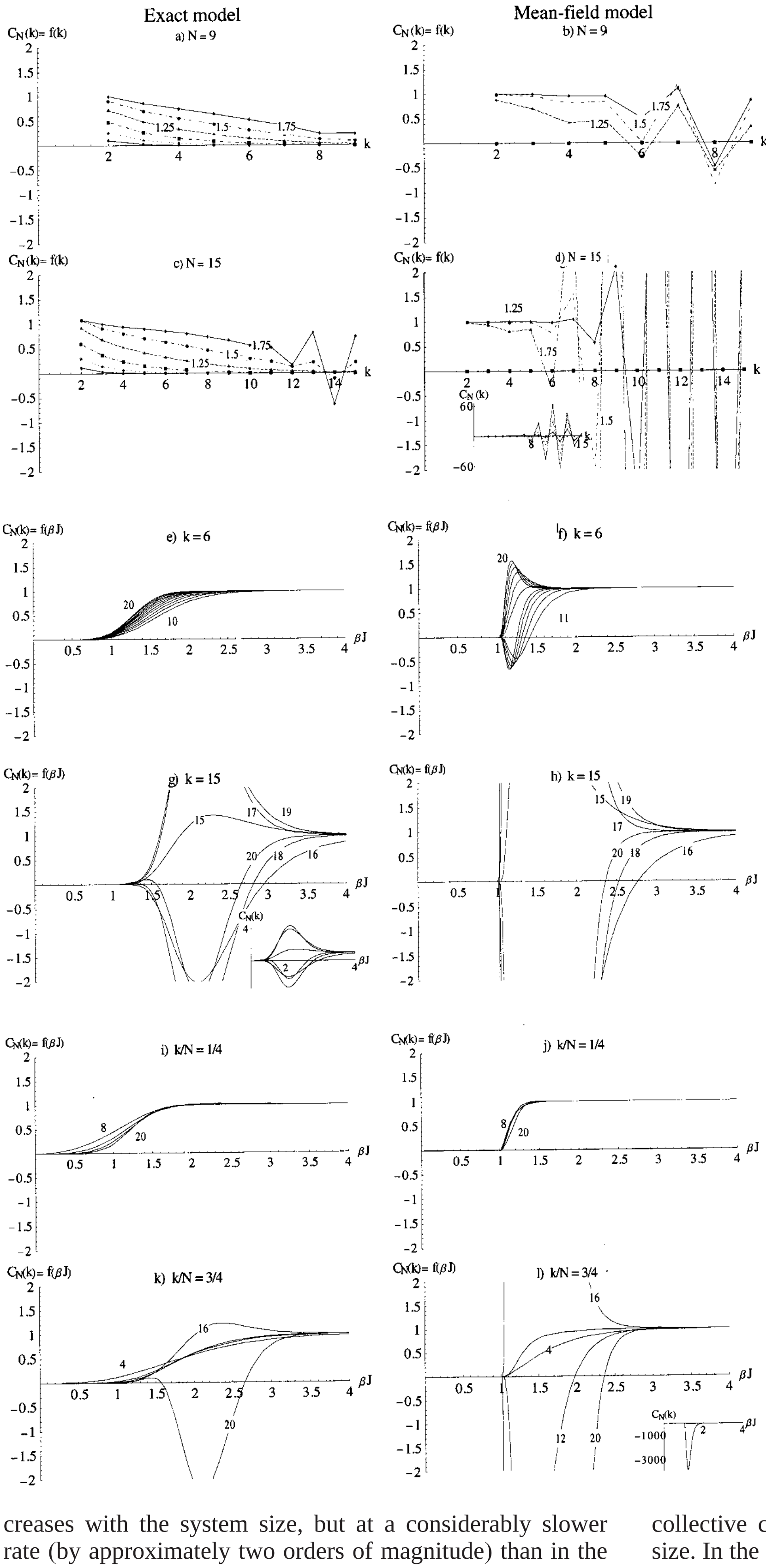

FIG. 2. (a)–(d) The complexities $C_N(k)$ of a system of $N=9$ and $N=15$ spins for increasing coupling strength $\beta J=0.5$, 0.75, 1.0, 1.25, 1.5, and 1.75, as a function of the scale k, $2 \leqslant k \leqslant N$. (e)–(h) Complexities $C_N(k)$ for a given scale $k=6$ and $k=15$ for systems with an increasing number of spins N, $10 \leqslant N \leqslant 20$, as a function of the coupling strength βJ. (i)–(l) Complexities $C_N(k)$ for given relative scale $k/N=1/4$ and $k=3/4$ in systems with at most 20 spins as a function of the coupling strength βJ.

creases with the system size, but at a considerably slower rate (by approximately two orders of magnitude) than in the mean-field case.

This feature has a twofold significance. First, oscillatory large-scale information correlations apparently are indicative of an ordered phase, or a coexistence of ordered phases, with collective constraints on a scale comparable to the system size. In the mean-field model, such a coexistence of phases is seen to develop abruptly above the transition point at $(\beta J)_{\rm cr}=1.0$, while in the exact model long-range correlations seem to build up gradually for $\beta J>1$, until a "constrained" behavior appears for $\beta J \geqslant 1.5$. Second, the growth of oscillations with the system size brings into question the existence of a thermodynamic limit for large-scale complexities. To explore this issue in more detail, in Figs. 2(e)–2(h) we examine complexities at a fixed scale as functions of the interaction strength, for increasing system sizes. Since in this case the minimum allowed N equals the selected scale k, the corresponding complexities for small systems describe large-scale correlations, whereas complexities for larger systems eventually describe fine-scale correlations. As the magnitude of the given scale increases, we expect the appearance of increasingly nonmonotonic complexities for the smaller systems, while larger systems should show the typical monotonic behavior characteristic of fine-scale correlations. This pattern is indeed observed for the exact model. The mean-field model, on the other hand, shows nonmonotonic behavior with the size of the system even at the relatively small scale $k=6$.

The origin of this phenomenon can be understood from the behavior of complexities as functions of coupling strength at a fixed scale-to-size ratio, constant k/N [Figs. 2(i)–2(l)]. The exact complexities display a clear converging trend with increasing system size for $k/N \leqslant 1/2$. At these scales we may infer the existence of a thermodynamic limit scaling as $C_{N\to\infty}(k) \approx C(k/N)$. For larger ratios, i.e., scales comparable to the system size, there is no apparent convergence, and the existence of a unique thermodynamic limit is doubtful. For the systems studied here, the divergence of complexities appears roughly for $\beta J \geqslant 1.5$. In the mean-field ferromagnetic phase, the complexities show convergence with the system size only for $k/N < 1/3$. At larger ratios, the divergent behavior begins abruptly above the critical point.

In both the exact and the mean-field model, the large-scale oscillations eventually disappear in the strong-coupling limit, and the complexities converge to the asymptotic unit value. This is more easily seen in Fig. 3, where complexities are plotted as functions of the relative scale k/N for increasing system sizes, at prescribed coupling strength [coupling strength increases in the panel sequence (a) to (f), (g) to (l), i.e., down the page, with the upper panels in the disordered phase and the lowest panels in the ordered phase]. The scaling behavior of fine-scale complexities is again evident. Regardless of the system size, the fine-scale complexities for a given k/N have virtually identical values, which increase smoothly from zero to one as the strength of the coupling increases. Figure 3 also gives a clear representation of the change of regime in the large-scale correlations of the exact model, in the neighborhood of $(\beta J)_0 \approx 1.5$. The discrepancy between this transition point and the mean-field critical point is a finite-size effect. The behavior of the critical point with increasing system size can be inferred from the corresponding finite-size specific heat (Fig. 4), which has its maximum in the vicinity of the same coupling value.

III. ANALYTIC DISCUSSION

The calculation of the numerical results presented in the previous section was considerably simplified by the permutation symmetry of the infinite-range Ising model, in both the exact and the mean-field version. For any configuration $\{s_i\}$ the corresponding energy and probability distribution $P(\{s_i\})$ are functionals only of the total magnetization of that configuration, or equivalently, of the number K of $\uparrow$ spins. The density of states at a specific energy is given by $\binom{N}{K}$. Similarly, the probability $\Sigma_{s_{i_1},s_{i_2},\ldots,s_{i_j}} P(\{s_i\})$ of a given configuration of $(N-j)$ spins with L $\uparrow$ spins, irrespective of the configuration of the remaining j spins, is a functional of L only, and not of the exact positions of the $(N-j)$ spins. Let us denote this probability by $P_{N-j;N}(L)$, and let $P_N(K)$ be the probability of an N-spin configuration with K $\uparrow$ spins, such that, under full permutation symmetry,

$$P_{N-j;N}(L) = \sum_{K=0}^{j} \binom{j}{K} P_N(K+L). \tag{6}$$

Recalling Eq. (1) for the complexities $C_N(k)$, we see that the sum of subsystem entropies $Q(N-j)$ can be calculated as

$$Q(N-j) = -\binom{N}{j} \sum_{L=0}^{N-j} \binom{N-j}{L} P_{N-j;N}(L) \log_2 P_{N-j;N}(L), \tag{7}$$

where the sum on the right-hand side gives the average entropy $\overline{Q}_N(N-j) = Q(N-j)/\binom{N}{N-j}$ of $(N-j)$ spins in the presence of j additional spins. Use of expression (7) in Eq. (1) yields the desired complexities $C_N(k)$. For the exact model, we have $P_N(K) = Z_N^{-1} \exp[(\beta J/2N)(N-2L-2K)^2]$, and the probabilities $P_{N-j;N}(L)$ are

$$P^0_{N-j;N}(L) = \frac{1}{Z_N} \sum_{K=0}^{j} \binom{j}{K} \exp\left[\frac{\beta J}{2N}(N-2L-2K)^2\right], \tag{8}$$

where the partition function is $Z_N = \Sigma_{k=0}^{N} \binom{N}{k} \exp[(\beta J/2N)(N-2k)^2]$.

As in a standard statistical calculation, the sums in Z_N and $P^0_{N-j}(L)$ may be given closed expressions by using a Gaussian transformation to linearize the quadratic exponents, and subsequently applying a steepest-descent approximation to the resulting integral [12]. But the steepest-descent approximation is valid only for large values of N and $N-2L$, respectively. The resulting expressions for $C(k)$, which involve sums over $L=0,1,\ldots,N$, yield good results only for fine scales, and break down above some scale $k < N/2$. These steepest-descent results are illustrated in Fig. 5. However, our reference calculations remain based on the exact expressions provided by Eqs. (7) and (8), and are compared to the mean-field approximation as a working framework, as described below and in Sec. IV.

We turn now to the analysis of the mean-field approximation. For the modified mean-field ansatz, the probability distribution is given by Eq. (4), and it is again a function only of the number K of spins $\uparrow$, $P^{(\mathrm{mf})}(\{s_i\}) \equiv P_N^{(\mathrm{mf})}(K)$. Also, due to

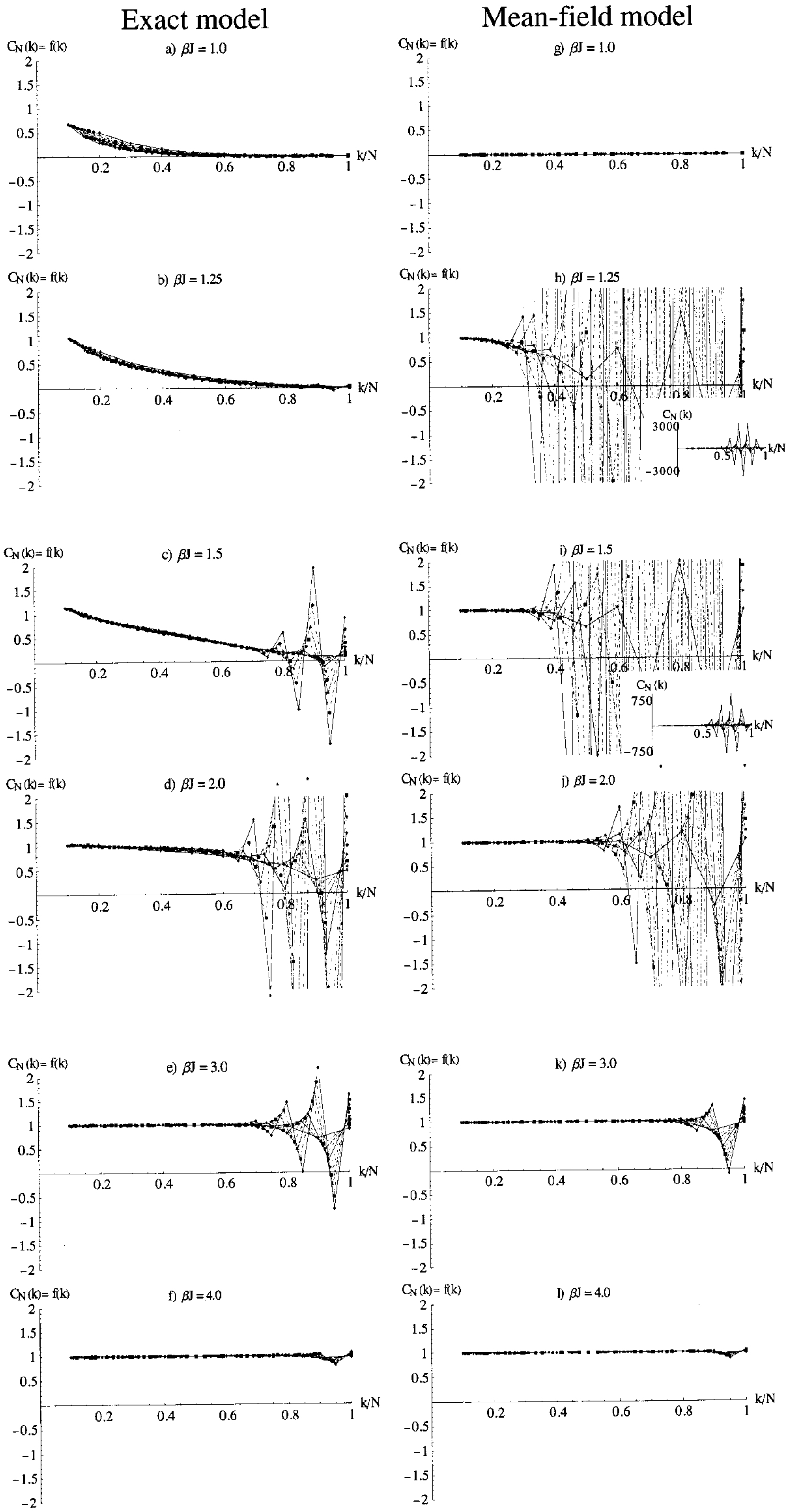

FIG. 3. The complexity $C_N(k)$ as a function of the relative scale k/N in systems with an increasing number of spins N, $10 \leqslant N \leqslant 20$, at given coupling strength $\beta J = 1.0$, 1.25, 1.5, 2.0, 3.0, and 4.0.

the product form of its two terms, the probability $P_{N-j;N}(L)$ for a configuration with $L\uparrow$ spins in a given set of $(N-j)$ spins is seen to be

$$P_{N-j;N}^{(\mathrm{mf})}(L) \equiv P_{N-j}^{(\mathrm{mf})}(L). \tag{9}$$

As a result, the average subsystem entropy $\overline{Q_N^{(\mathrm{mf})}}(N-j)$ is identical to the mean-field entropy $S_{N-j}^{(\mathrm{mf})}$ for a system of $(N-j)$ spins (i.e., in the absence of any additional spins).

It is worth noting that the entropy $S_N^{(\mathrm{mf})}$ for the modified distribution (4) deviates slightly from the standard mean-field entropy. We show in the Appendix that this deviation is always less than one bit in absolute value, hence it becomes

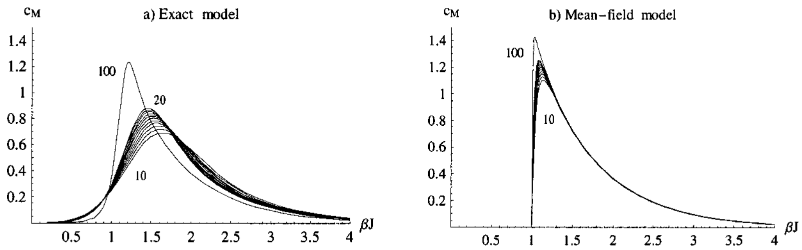

FIG. 4. The specific heat c_M as a function of the coupling strength βJ. (a) Exact model, (b) modified mean-field model.

negligible in the thermodynamic limit. Nonetheless, it is this small, nonextensive deviation that accounts for the occurrence of nontrivial information correlations in the modified mean-field model. To see this, it suffices to express the complexities (1) in terms of the average subsystem entropies,

$$C_N^{(\mathrm{mf})}(k) = k\binom{N}{k}\sum_{j=0}^{k-1}(-1)^{k-1-j}\binom{k-1}{j}\frac{S_{N-j}^{(\mathrm{mf})}}{N-j},$$

and observe that the sum on the right-hand side vanishes for an extensive entropy $S_{N-j}^{(\mathrm{mf})} \sim (N-j)$, unless $k=1$.

We write now, employing the results in the Appendix [Eq. (A4)],

$$S_N^{(\mathrm{mf})} = Ns^{(\mathrm{mf})} + 1 - \Delta S_N^{(\mathrm{mf})}, \tag{10}$$

where $s^{(\mathrm{mf})}$ is the usual mean-field entropy per spin, and $\Delta S_N^{(\mathrm{mf})} \approx (1-m^2)N/2$ [Eqs. (A3) and (A7)]. Using this approximation in the previous expression for $C_N^{(\mathrm{mf})}(k)$, together with the identity $k\binom{N}{K}\sum_{j=0}^{k-1}(-1)^{k-1-j}\binom{k-1}{j}(N-j)^{-1} \equiv 1$, yields

$$C_N^{(\mathrm{mf})}(k) \approx \delta_{k,1}Ns^{(\mathrm{mf})} + 1 - k\binom{N}{k}\sum_{j=0}^{k-1}(-1)^{k-1-j} \times \binom{k-1}{j}\frac{(1-m^2)(N-j)/2}{N-j}. \tag{11}$$

The last two terms in the expression on the right-hand side may be recognized as an incomplete beta function [13], and we are left with the following closed-form approximation for the mean-field complexities:

$$C_N^{(\mathrm{mf})}(k) \approx \delta_{k,1}Ns^{(\mathrm{mf})} + I_{1-\sqrt{1-m^2}}(k, N-k+1). \tag{12}$$

The domain of validity of expression (12) is restricted by the approximation for the terms $\Delta S_{N-j}^{(\mathrm{mf})}$ to $(N-j) \gg 1$, which implies both a large total system ($N \gg 1$) and a small relative scale, $k/N \ll 1$. Thus the characteristic large-scale oscillations above the transition point are not captured by this approximation. In fact, from the properties of the incomplete beta function, it can be seen that

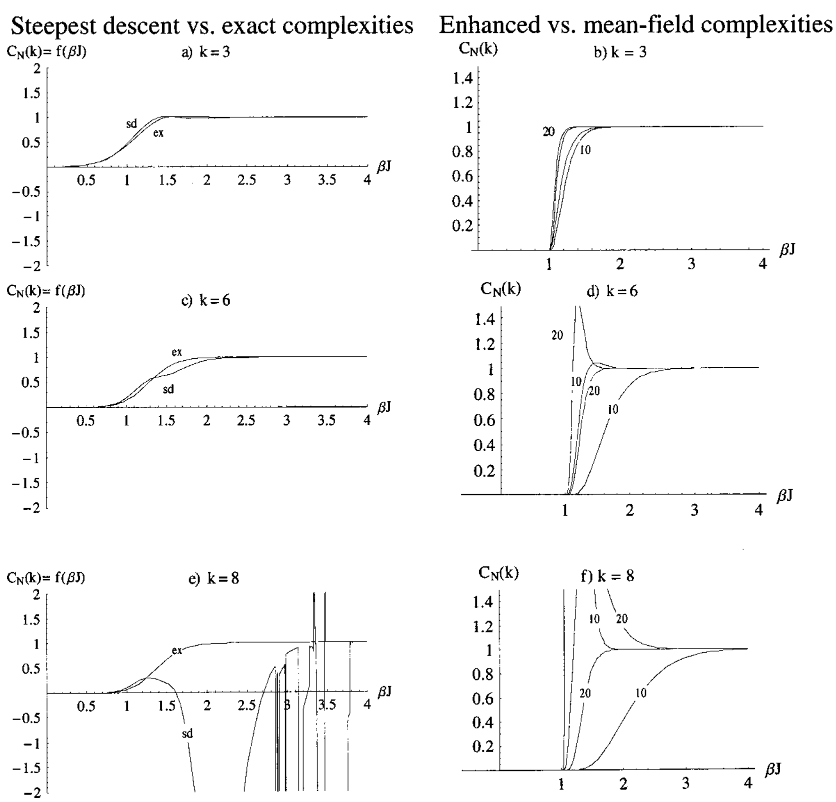

FIG. 5. Complexities as functions of the coupling strength βJ at a given scale $k=3$, 6, 8 for a system with $N=20$ spins: (a),(c),(e) exact vs steepest-descent complexities; (b),(d),(f) mean-field complexities and their approximation by Eq. (12).

$$0 \leqslant \mathcal{C}_N^{(\mathrm{mf})}(k) \leqslant 1$$

for any scale $k > 1$. The approximation Eq. (12) and the actual mean-field complexity are compared in Fig. 5.

IV. FLUCTUATIONS BEYOND THE MEAN-FIELD APPROXIMATION

Although the mean-field results capture many fundamental features of the exact information correlations, it is evident that they do not provide good quantitative approximations to the latter. A better description of scale-dependent correlations demands a statistical description beyond the simple mean-field ansatz. We now show that it is possible to augment the modified mean-field model to improve the agreement for both fine-scale and large-scale correlations.

The main idea is to regard the mean-field magnetization as a stochastic parameter with a Gaussian distribution. We assume that the system samples randomly mean-field distributions of type (4), each distribution occurring with a prescribed probability. Further, we assume that this probability has a Gaussian dependence on the magnetization μ, e.g., $A\ \exp[-a(m_G-\mu)^2]$, where A is a normalization factor, m_G sets the center of the Gaussian distribution, and a determines the width of the distribution. Here all parameters must be regarded as functionals of the coupling constant βJ. We set m_G to the mean-field magnetization given by Eq. (5), $m_G \equiv m$, and let the width a be determined such that the root-mean-square deviation of the magnetization, $\langle \Delta(\Sigma_i s_i)^2 \rangle^{1/2}$, is identical to the magnetization deviation in the exact model. The corresponding probability distribution for a spin configuration with $K\uparrow$ spins becomes

$$P_N^{(\mathrm{mf})}(K) = A \int_{-1}^{1} d\mu\ \exp[-a(m-\mu)^2] \times \frac{1}{2}\left[\left(\frac{1+\mu}{2}\right)^k \left(\frac{1-\mu}{2}\right)^{N-k} + \left(\frac{1+\mu}{2}\right)^{N-k} \left(\frac{1-\mu}{2}\right)^k\right]. \quad (13)$$

with A a normalization constant, and the Gaussian width a is a solution of

$$N + N(N-1)\frac{\int_{-1}^{1} d\mu\ \mu^2 \exp[-a(m-\mu)^2]}{\int_{-1}^{1} d\mu\ \exp[-a(m-\mu)^2]} = \frac{1}{Z_N}\sum_{k=0}^{N}\binom{N}{k}(N-2k)^2 \exp\left[\frac{J}{2N}(N-2k)^2\right]. \quad (14)$$

Note that the mean-field distribution averaged in Eq. (13) is retained in symmetric form with respect to the magnetization μ, in order to account for the two possible orientations of the ordered phase and the corresponding macroscopic bit of information. For illustration, we apply the above enhanced model to a system of 20 spins. In this case, we find that a good approximation to the solution of Eq. (14) is given by (see Fig. 6)

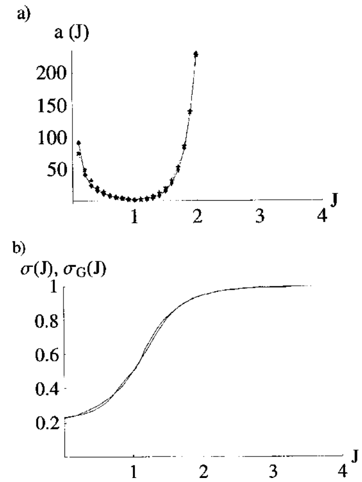

FIG. 6. (a) Exact and approximate solutions of Eq. (14) for the Gaussian width $a(\beta J)$ against the coupling βJ. The approximate solution is defined by Eq. (15). (b) Square-root deviation of the magnetization per spin against the coupling βJ in the exact model $[\sigma(\beta J)]$ and in the enhanced model with Gaussian width given by Eq. (15) $[\sigma_G(\beta J)]$.

$$a(\beta J) \approx \begin{cases} 1.69\ \exp[-4.2(J-1.0)] \text{ for } J < 1.0, \\ 1.69\ \exp[4.9(J-1.0)] \text{ for } J \geqslant 1.0. \end{cases} \quad (15)$$

Figure 6(b) shows that the agreement between the square-root deviation of the magnetization in the exact and in the present model with the approximation given by Eq. (15) is indeed very good. Comparative plots of complexities at selected scales are presented in Fig. 7. As an immediate effect of the Gaussian averaging, the enhanced model no longer displays a well-defined phase transition, and all correlations have a smooth dependence on the coupling constant, as in the exact results. For $\beta J < 1.0$, there are only small-amplitude fine-scale correlations that vanish as the coupling decreases, whereas for $\beta J > 1.0$ the system gradually develops large-scale correlations. We note that the quantitative agreement of fine-scale correlations ($k \leqslant 20$) becomes remarkably good over the entire interaction range studied. For large-scale correlations, the nonmonotonic behavior characteristic of collective "constraints" is still seen at lower coupling, lower scales, and with larger amplitudes, than in the exact model. This is shown in detail in Figs. 7(e)–7(n), which compare plots of complexities against relative scale at a given coupling. However, there is a significant decrease in the amplitude of the associated maxima and minima as compared to the mean-field model, and the agreement with the exact model is also improved in the strong-coupling limit, as the scale-dependent oscillations diminish [Figs. 7(h), 7(i), 7(m), and 7(n)]. We note again that in the strong-coupling limit, all complexities approach 1, and thus all information about the system is in the macroscopic bit due to the two alternative orientations of the ordered phase.

V. CONCLUSIONS

Our study of k-fold correlations in the infinite-range, ferromagnetic Ising model reveals a contrast in the behavior of

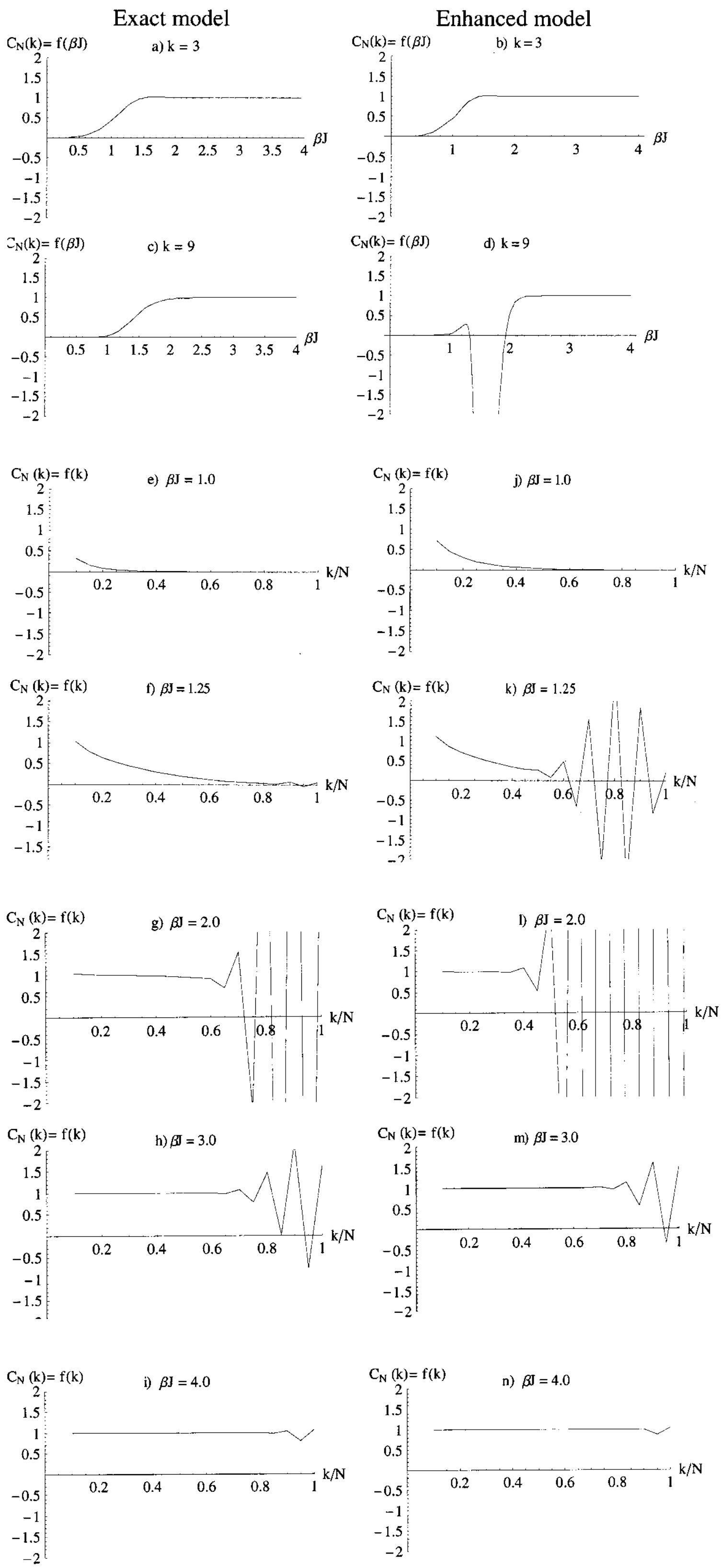

FIG. 7. (a)–(d) The complexities $C_N(k)$ of a system of $N=20$ spins for selected scales $k=3$ and $k=9$, as a function of the coupling strength βJ. Left panels (a), (c) show the exact model; right panels (b), (d) show the enhanced mean-field model. (e)–(n) The complexity $C_N(k)$ as a function of the relative scale k/N in a system with $N=20$ spins, at given coupling strength βJ $=1.0, 1.25, 2.0, 3.0,$ and 4.0. Left panels (e)–(i) show the exact model; right panels (j)-(n) show the enhanced mean-field model.

fine-scale and large-scale information interdependencies with increasing coupling strength and system size. We found that in both the exact model and the mean-field version, the fine-scale correlations have a smooth, mostly monotonic variation that converges with increasing system size towards a well-defined thermodynamic limit, both below and above the transition point. They also display a clear scaling behavior with the relative scale k/N, and for a fixed ratio $k/N<1/2$ ($k/N<1/3$ in mean-field) are virtually independent of the system size. Large-scale correlations have a qualitatively similar behavior below the transition point, in the disordered phase, whereas above the transition point they oscillate with amplitudes that increase with the size of the system. This feature indicates a nonanalytic behavior in the thermodynamic limit. It is also reminiscent of the typical nonmonotonic behavior of the usual statistical correlation function in the vicinity of the critical point. Eventually, the large-scale oscillations diminish and disappear in the strong-coupling limit, when the scale-specific amount of information becomes independent of the scale itself. This study lays the groundwork for applications of information theoretic measures to models of traditional and nontraditional statistical mechanical systems.

ACKNOWLEDGMENT

The authors are grateful to Professor Kardar Mehran for his helpful comments and suggestions on this manuscript.

APPENDIX: THE MODIFIED MEAN-FIELD ENTROPY

In this appendix, we show that the total entropy for the modified mean-field distribution (4),

$$\mathcal{S}_N^{(\mathrm{mf})} = -\sum_{k=0}^{N}\binom{N}{k}P_N^{(\mathrm{mf})}(k)\log_2 P_N^{(\mathrm{mf})}(k), \tag{A1}$$

differs from the standard mean-field entropy $Ns^{(\mathrm{mf})}$ by a nonextensive amount that is less than one bit in absolute value, and we find a suitable analytic approximation for this deviation.

We start by bringing the entropy (A1) to the form

$$\mathcal{S}_N^{(\mathrm{mf})} = -\sum_{K=0}^{N}\binom{N}{K}\frac{1}{2}\left(\frac{1+m}{2}\right)^N f\left(\frac{1-m}{1+m};N,K\right) \times \log_2\left[\frac{1}{2}\left(\frac{1+m}{2}\right)^N f\left(\frac{1-m}{1+m};N,K\right)\right], \tag{A2}$$

where $f(x;N,K)=x^{N-K}+x^K$. The sum over the term $\log_2\{(1/2)[(1+m)/2]^N\}$ reduces to 1, while the sum containing the remaining logarithmic factor can be rearranged as

$$-\frac{1}{2}\left(\frac{1+m}{2}\right)^N\sum_{K=0}^{N}\binom{N}{K}f_{N,K}\left(\frac{1-m}{1+m}\right)\log_2 f_{N,K}\left(\frac{1-m}{1+m}\right) = N_{\mathcal{S}}^{(\mathrm{mf})} - \Delta\mathcal{S}_N^{(\mathrm{mf})},$$

where $s^{(\mathrm{mf})}=1-\frac{1}{2}[(1+m)\log_2(1+m)+(1-m)\log_2(1-m)]$ is the usual mean-field entropy per spin, and

$$\Delta\mathcal{S}_N^{(\mathrm{mf})} = \left(\frac{1+m}{2}\right)^N\sum_{K=0}^{N}\binom{N}{K}\left(\frac{1-m}{1+m}\right)^K \times\log_2\left[1+\left(\frac{1-m}{1+m}\right)^{N-2K}\right]. \tag{A3}$$

This yields the total entropy in the form

$$\mathcal{S}_N^{(\mathrm{mf})} = N_{\mathcal{S}}^{(\mathrm{mf})} + 1 - \Delta\mathcal{S}_N^{(\mathrm{mf})}. \tag{A4}$$

Note that the unit term in expression (A4) represents a macroscopic bit due to the coexistence in the modified ansatz of ordered phases with opposite magnetizations.

To see that the overall correction $(1-\Delta\mathcal{S}_N^{(\mathrm{mf})})$ is always less than one bit in magnitude, it suffices to recall the well-known inequality $\ln(1+x)=\ln 2\,\log_2(1+x)\leqslant x$, for any $x>-1$, and to observe that

$$0 \leqslant \Delta\mathcal{S}_N^{(\mathrm{mf})} \leqslant \frac{1}{\ln 2}\left(\frac{1+m}{2}\right)^N\sum_{K=0}^{N}\binom{N}{K}\left(\frac{1-m}{1+m}\right)^K\left(\frac{1-m}{1+m}\right)^{N-2K} = \frac{1}{\ln 2}. \tag{A5}$$

Hence

$$|1-\Delta\mathcal{S}_N^{(\mathrm{mf})}| \leqslant 1. \tag{A6}$$

A first-order approximation for $\Delta\mathcal{S}_N^{(\mathrm{mf})}$ follows if we note that the factor $[(1-m)/(1+m)]^K\log_2\{1+[(1-m)/(1+m)]^{N-2K}\}$ under the sum in Eq. (A3) attains a maximum at some K_0 such that $[(1-m)/(1+m)]^{N-2K_0}\approx 3.923$. In this case $[(1-m)/(1+m)]^{K_0}\approx(1/2)[(1-m)/(1+m)]^{N/2}$, and every term of the sum is less than or equal to

$$\alpha\binom{N}{K}[(1-m)/(1+m)]^{N/2},$$

where $\alpha\sim 1.161$. For a crude estimate, we approximate the entire sum as

$$\sum_{K=0}^{N}\binom{N}{K}\left(\frac{1-m}{1+m}\right)^K\log_2\left[1+\left(\frac{1-m}{1+m}\right)^{N-2K}\right] \approx \left(\frac{1-m}{1+m}\right)^{N/2}2^N,$$

and obtain

$$\Delta\mathcal{S}_N^{(\mathrm{mf})} \approx (1-m^2)^{N/2}. \tag{A7}$$

It also follows, from the corresponding upper bound, that $\Delta\mathcal{S}_N^{(\mathrm{mf})}$ vanishes as $N\to\infty$ for $m>0$.

[1] J. J. Binney, N. J. Dowrick, A. J. Fisher, and M. E. J. Newman, *The Theory of Critical Phenomena* (Clarendon, Oxford, 1992); D. A. Lavis and G. M. Bell, *Statistical Mechanics of Lattice Systems*, Vol. 2: Exact, Series and Renormalization Group Methods (Springer-Verlag, Berlin, 1999).

[2] C. E. Shannon and W. Weaver, *The Mathematical Theory of Communication* (University of Illinois Press, Urbana, 1963).

[3] D. J. Amit, *Modeling Brain Function: The World of Attractor Neural Networks* (Cambridge University Press, Cambridge, UK, 1989); Y. S. Abu-Mostafa, IEEE Commun. Mag. **27**, 25

(1989); J. Taylor and M. Plumbley, *Information Theory and Neural Networks, in Mathematical Approaches to Neural Networks*, edited by J. Taylor (Elsevier Science, Amsterdam, 1993), p. 307; G. Tononi, O. Sporns, and G. Edelman, Proc. Natl. Acad. Sci. U.S.A. **91**, 5033 (1994); G. Deco and D. Obradovic, *An Information-Theoretic Approach to Neural Computing* (Springer-Verlag, New York, 1996); O. A. Rosso, Physica A **313**, 587 (2002); D. MacKay, *Information Theory, Inference and Learning Algorithms* (Cambridge University Press, Cambridge, UK, 2003).

[4] P. Grassberger, Int. J. Theor. Phys. **25**, 907 (1986); K. Lindgren and M. G. Norhdal, Complex Syst. **2**, 409 (1988); *Cellular Automata and Modeling of Complex Systems*, Vol. 46 of *Springer Proceedings in Physics*, edited by P. Manneville, N. Boccara, G. I. Vichniac, and R. Bideaux (Springer, Berlin, 1990); *Cellular Automata: Theory and Experiment*, edited by H. A. Gutowitz (MIT Press, Cambridge, MA, 1991).

[5] P. Szepfalusy and G. Gyorgyi, Phys. Rev. A **33**, 2852 (1986); N. B. Abraham, A. M. Albano, A. Passamante, and P. E. Rapp, *Measures of Complexity and Chaos* (Plenum, New York, 1989); L. Diambra and A. Plastino, Phys. Lett. A **216**, 278 (1996); J. Freund, W. Ebeling, and K. Rateitschak, Phys. Rev. E **54**, 5561 (1996); Y.-C. Tian, Physica D **108**, 113 (1997); C. Schittenkopf, *ibid.* **110**, 173 (1997); A. G. Bashkirov and A. V. Vityazev, Physica A **277**, 136 (2000); G. Boffetta, Phys. Rep. **356**, 367 (2002); A. Shabunin, V. Demidov, V. Astakhov, and V. Anishchenko, Phys. Rev. E **65**, 056215 (2002); C. Beck and F. Schlogl, *Thermodynamics of Chaotic Systems* (Cambridge University Press, Cambridge, UK, 1993); R. Metzler, Y. Bar-Yam, and M. Kardar, Phys. Rev. E **70**, 026205 (2004).

[6] C. H. Bennett, Found. Phys. **16**, 585 (1986); K. Lindgren, Phys. Rev. A **38**, 4794 (1988); *Complexity, Entropy and the Physics of Information*, edited by W. Zurek (Addison-Wesley, Redwood City, CA, 1990); R. Wackerbauer, A. Witt, H. Atmanspacher, J. Kurths, and H. Scheingraber, Chaos, Solitons Fractals **4**, 133 (1994); Y. Bar-Yam, *Dynamics of Complex Systems* (Addison-Wesley, Reading, MA, 1997); R. Badii and A. Politi, *Complexity: Hierarchical Structure and Scaling in Physics* (Cambridge University Press, Cambridge, UK, 1997); J. S. Shiner, M. Davison, and P. T. Landsberg, Phys. Rev. E **59**, 1459 (1999); Yu. A. Andrienko, N. V. Brilliantov, and J. Kurths, Eur. Phys. J. B **15**, 539 (2000); P. M. Binder and J. A. Plazas, Phys. Rev. E **63**, 065203(R) (2001).

[7] J. P. Crutchfield and D. P. Feldman, Phys. Rev. E **55**, R1239 (1997). They describe the effect of ordering on information in a one-dimensional Ising model.

[8] Y. Bar-Yam, Complexity **9**, 37 (2004); **9**, 15 (2004).

[9] Y. Bar-Yam, Adv. Complex Syst. **7**, 47 (2004).

[10] D. R. Wolf, Ph. D. dissertation, University of Texas, Austin, Texas, 1996.

[11] H. S. Green, *The Molecular Theory of Fluids* (North-Holland, Amsterdam, 1952).

[12] S. Gheorghiu-Svirschevski and Y. Bar-Yam (unpublished).

[13] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. Stegun (Dover, New York, 1970).